

N=2 supersymmetric unconstrained matrix GNLS hierarchies are consistent

F. Delduc^{a,1}, O. Lechtenfeld^{b,2}, and A.S. Sorin^{c,3}

^(a) *Laboratoire de Physique[†], Groupe de Physique Théorique,
ENS Lyon, 46 Allée d'Italie, 69364 Lyon, France*

^(b) *Institut für Theoretische Physik, Leibniz Universität Hannover,
Appelstraße 2, D-30167 Hannover, Germany*

^(c) *Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
141980 Dubna, Moscow Region, Russia*

Abstract

We develop a pseudo-differential approach to the $N=2$ supersymmetric unconstrained matrix $(k|n, m)$ -Generalized Nonlinear Schrödinger hierarchies and prove consistency of the corresponding Lax-pair representation (nlin.SI/0201026). Furthermore, we establish their equivalence to the integrable hierarchies derived in the super-algebraic approach of the homogeneously-graded loop superalgebra $sl(2k+n|2k+m) \otimes C[\lambda, \lambda^{-1}]$ (nlin.SI/0206037). We introduce an unconventional definition of $N=2$ supersymmetric strictly pseudo-differential operators so as to close their algebra among themselves.

PACS: 02.20.Sv; 02.30.Jr; 11.30.Pb

Keywords: Completely integrable systems; Supersymmetry; Discrete symmetries

1) *E-Mail:* francois.delduc@ens-lyon.fr

2) *E-Mail:* lechtenf@itp.uni-hannover.de

3) *E-Mail:* sorin@theor.jinr.ru

†) UMR 5672 du CNRS, associée à l'Ecole Normale Supérieure de Lyon.

1 Introduction and Summary

The $N=2$ supersymmetric unconstrained matrix $(k|n, m)$ -Generalized Nonlinear Schrödinger $((k|n, m)$ -GNLS) hierarchies were proposed in [1] by exhibiting the corresponding *matrix pseudo-differential* Lax-pair representation

$$L = I\partial + F D \bar{D} \partial^{-1} \bar{F}$$

in terms of a $k \times (m+n)$ matrix F with $N=2$ *unconstrained superfield* entries for the bosonic isospectral flows. Their super-algebraic formulation and recursion relations were proposed in [2] on the basis of the homogeneously-graded loop superalgebra $sl(2k+n|2k+m) \otimes C[\lambda, \lambda^{-1}]$.

These hierarchies generalize and contain as limiting cases many other interesting $N=2$ supersymmetric hierarchies discussed in the literature: When matrix entries are chiral and antichiral $N=2$ superfields, these hierarchies reproduce the $N=2$ chiral matrix $(k|n, m)$ -GNLS hierarchies [3, 4], and in turn the latter coincide with the $N=2$ GNLS hierarchies of references [5, 6] in the scalar case $k=1$. When matrix entries are unconstrained $N=2$ superfields and $k=1$, these hierarchies are equivalent to the $N=2$ supersymmetric multicomponent hierarchies [7]. The bosonic limit of the $N=2$ unconstrained matrix $(k|0, m)$ -GNLS hierarchy reproduces the bosonic matrix NLS equation elaborated in [8] via the $gl(2k+m)/(gl(2k) \times gl(m))$ -coset construction. The $N=2$ matrix $(1|1, 0)$ -GNLS hierarchy is related to one of three different existing $N=2$ supersymmetric KdV hierarchies – the $N=2$ $\alpha=1$ KdV hierarchy – by a reduction [7, 1, 9].

Self-consistency of the Lax-pair representation for the $N=2$ supersymmetric unconstrained matrix $(k|n, m)$ -GNLS hierarchies was actually proven in [1] only for the first four flows, but conjectured for the general case. The equivalence of their super-algebraic and pseudo-differential formulations was established in [2], but again for the first few flows only. The present letter completes these proofs. In Section 2 we develop a pseudo-differential approach to the $N=2$ supersymmetric unconstrained matrix $(k|n, m)$ -GNLS hierarchies in $N=2$ super-space and rigorously construct their Lax-pair representation

$$\frac{\partial}{\partial t_p} L = [(L^p)_\oplus, L]$$

with

$$\frac{\partial}{\partial t_p} F = \left((L^p)_\oplus F \right) \quad \text{and} \quad \frac{\partial}{\partial t_p} \bar{F} = -\left(\bar{F} (\bar{L}^p)_\oplus \right),$$

where we introduce an unconventional definition of $N = 2$ supersymmetric strictly pseudo-differential operators so as to close their algebra among themselves¹. Furthermore, we produce the recursion relations for the corresponding isospectral flows. As we establish in Section 3, this Lax-pair representation agrees with the one derived in the super-algebraic approach of the homogeneously-graded loop superalgebra $sl(2k+n|2k+m) \otimes C[\lambda, \lambda^{-1}]$. Thus, we finally prove the conjectured equivalence of the two hierarchies.

Apart from the Lax-pair representation for the isospectral flows and the recursion relations of these hierarchies, we presently do not know other characteristic properties like their (bi)Hamiltonian structures, discrete symmetries etc., although part of these are known for some limiting cases. We hope to address these problems in future.

¹The precise notation will be explained there.

2 Pseudo-differential approach

Our starting point is the Lax operator for the $N=2$ supersymmetric unconstrained matrix $(k|n, m)$ -GNLS hierarchies introduced in [1]²

$$L = I\partial + F D \bar{D} \partial^{-1} \bar{F}. \quad (1)$$

Here, $F \equiv F_{Aa}(Z)$ and $\bar{F} \equiv \bar{F}_{aA}(Z)$ ($A, B = 1, \dots, k$; $a, b = 1, \dots, n+m$) are rectangular matrices whose entries are unconstrained $N = 2$ superfields, I is the unity matrix, $I_{AB} \equiv \delta_{AB}$, and the matrix product is implied, for example $(F\bar{F})_{AB} \equiv \sum_{a=1}^{n+m} F_{Aa} \bar{F}_{aB}$ and $(F\bar{F})_{ab} \equiv \sum_{A=1}^k F_{aA} \bar{F}_{Ab}$. The matrix entries are Grassmann even superfields for $a = 1, \dots, n$ and Grassmann odd superfields for $a = n+1, \dots, n+m$. Thus, fields do not commute, but rather satisfy $F_{Aa} \bar{F}_{bB} = (-1)^{d_a \bar{d}_b} \bar{F}_{bB} F_{Aa}$ where d_a and \bar{d}_b are the Grassmann parities of the matrix elements F_{Aa} and \bar{F}_{bB} , respectively, $d_a = 1$ ($d_a = 0$) for odd (even) entries. Superfields depend on the coordinates $Z = (z, \theta, \bar{\theta})$ of $N = 2$ superspace, and the $N = 2$ supersymmetric fermionic covariant derivatives D, \bar{D} are

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = -\frac{\partial}{\partial z} \equiv -\partial. \quad (2)$$

The set of operators

$$\{\hat{o}_i, i \in \mathbb{Z}\} := \{D^n \bar{D}^{\bar{n}} \partial^m, n, \bar{n} = 0, 1, m \in \mathbb{Z}\} \quad (3)$$

forms a basis in the associative algebra of the supermatrix valued pseudo-differential operators on $N = 2$ superspace

$$\mathbb{O} = \sum_{i=-\infty}^{\infty} f_i \hat{o}_i := \sum_{m=-\infty}^{N_{max}} \sum_{n, \bar{n}=0}^1 f_{n, \bar{n}, m} D^n \bar{D}^{\bar{n}} \partial^m, \quad d_{\mathbb{O}} = d_{\hat{o}_i} + d_{f_i} \quad (4)$$

where f_i is a supermatrix valued $N = 2$ superfield and \hat{o}_i is a basis operator with the Grassmann parities d_{f_i} and $d_{\hat{o}_i}$, respectively, and we understand that the operator \mathbb{O} possesses a definite Grassmann parity $d_{\mathbb{O}}$. We shall say that \mathbb{O} above is a differential operator if the sum over m is restricted to positive or zero values only, and that \mathbb{O} is strictly pseudo-differential if the sum over m is restricted to negative values of m . The set of the differential operators and the set of the strictly pseudo-differential operators both form a subalgebra of the whole space of pseudo-differential operators. Any pseudo-differential operator \mathbb{O} is the sum of a differential operator \mathbb{O}_{\oplus} and a strictly pseudo-differential operator \mathbb{O}_{\ominus} . Hereafter, the notation $(L^p)_{\hat{o}_i}$ denotes the supermatrix coefficient of the basis element \hat{o}_i , i.e. $(L^p)_{\hat{o}_i} \hat{o}_i$ belongs to the expansion (4) of L^p ; $(\mathbb{O}f)$ has the meaning of a supermatrix valued pseudo-differential operator \mathbb{O} acting only on a supermatrix valued function f inside the brackets³.

²Note, the supermatrices $\{F, \bar{F}\}$ are re-scaled by $\sqrt{2}$ comparing to [1].

³In order to avoid misunderstanding, let us remark the difference in the notations $(\mathbb{O}f)$ and $\{(\mathbb{O}f)_{\oplus}, (\mathbb{O}f)_{\ominus}\}$: the former represents a supermatrix valued superfield, while the latter corresponds to a differential and strictly pseudo-differential parts of the operator $\mathbb{O}f$, respectively.

Remark. The definition which is used in this paper of a strictly pseudo-differential operator slightly differs from the one which was used in the articles [1, 2]. There, instead of the basis elements $D\overline{D}\partial^n$, one was rather using as basis elements the operators $[D, \overline{D}]\partial^n$. Differential and strictly pseudo-differential operators were defined with respect to this basis. Any pseudo-differential operator \mathbb{O} may be separated into a differential operator \mathbb{O}_+ and a strictly pseudo-differential operator \mathbb{O}_- according to this basis. This other definition has the drawback that strictly pseudo-differential operators do not form a closed algebra, because of the relation $([D, \overline{D}]\partial^{-1})^2 = 1$. The relation between the two definitions is easily obtained by using the relation $D\overline{D}\partial^{-1} = -\frac{1}{2} + \frac{1}{2}[D, \overline{D}]\partial^{-1}$. Defining the residue of the operator \mathbb{O} as the coefficient of $D\overline{D}\partial^{-1}$ in the new basis (this differs by a factor of 2 from the definition in [1, 2], where it was defined as the coefficient of $[D, \overline{D}]\partial^{-1}$), one finds the following relation

$$\mathbb{O}_\oplus = \mathbb{O}_+ + \frac{1}{2} \text{res } \mathbb{O}, \quad (5)$$

which allows one to relate calculations in previous articles and in the present article.

Definition 1. We define the involutive automorphism $*$ of the second order of the supersymmetry algebra

$$\{\partial, D, \overline{D}\}^* = -\{\partial, D, \overline{D}\}, \quad (D\overline{D})^* = -\overline{D}D, \quad (\overline{D}D)^* = -D\overline{D}. \quad (6)$$

It can be extended to all basis elements in (3) using the rule $(\hat{o}_i \hat{o}_j)^* = (-1)^{d_{\hat{o}_i} d_{\hat{o}_j}} \hat{o}_j^* \hat{o}_i^*$. When applied to a supermatrix $f \Rightarrow f^*$ simply amounts to a change in the sign of its Grassmann-odd entries. Its k -fold action on the supermatrix f will be denoted $f^{*(k)}$ ($k = 0, 1 \bmod 2$).

Remark. Relations (6) reproduce the conventional operator-conjugation rules for the fermionic and bosonic covariant derivatives, although the star-operation $*$, being applied to a supermatrix f^* , differs comparing to the conventional operation of the super-transposition of a supermatrix f^T , i.e. $f^* \neq f^T$. We also remark that $(F\overline{F})^* \equiv F^* \overline{F}^* = F\overline{F}$, $F\overline{F}^* = F^* \overline{F}$, while $(\overline{F}F)^* \equiv \overline{F}^* F^* \neq \overline{F}F$.

Definition 2. We introduce the adjoint operator $\overleftarrow{\mathbb{O}}$ by defining its action on the supermatrix valued superfield f with the Grassmann parity d_f

$$f \overleftarrow{\mathbb{O}} := \sum_{i=-\infty}^{\infty} \hat{o}_i^* (f f_i)^{*(d_{\hat{o}_i})}. \quad (7)$$

Remark. This definition generalizes the definition of the adjoint operator to the non-abelian, noncommutative case. For the abelian, commutative case, i.e. when f and f_i are not (super)matrices, but commutative functions, it reproduces the conventional definition of the adjoint operator. Due to this reason we call the operator $\overleftarrow{\mathbb{O}}$ noncommutatively-adjoint operator.

Equation (7) defines a product of a noncommutatively-adjoint operator and a supermatrix-valued superfield. In order to consistently define a product of different noncommutatively-adjoint operators with themselves, we firstly need to prove:

Proposition 1.

$$(f \overleftarrow{\mathbb{O}_1 \dots \mathbb{O}_k}) = (f \overleftarrow{\mathbb{O}_1} \dots \overleftarrow{\mathbb{O}_k}) \equiv (((f \overleftarrow{\mathbb{O}_1}) \dots) \overleftarrow{\mathbb{O}_k}). \quad (8)$$

Proof. By induction, it is sufficient to check (8) for two operators \mathbb{O}_1 and \mathbb{O}_2 , which has been done with the help of the rules (6,7). ■

Remark. Proposition 1 gives the result of the action of a product of noncommutatively-adjoint operators on a supermatrix-valued superfield. Using the latter we define a product $\prod_{i=1}^k \overleftarrow{\mathbb{O}_i}$ of noncommutatively-adjoint operators which generalizes the definition of the product of the conjugated operators in the commutative case to the noncommutative case.

Definition 3.

$$f \overleftarrow{\mathbb{O}_1} \dots \overleftarrow{\mathbb{O}_k} := f \overleftarrow{\mathbb{O}_1 \dots \mathbb{O}_k}. \quad (9)$$

Remark. It is obvious that the r.h.s. of eq. (9) can be calculated using eq. (7), if one takes into account that due to the associativity of the algebra of pseudo-differential operators the product $\mathbb{O}_1 \dots \mathbb{O}_k = \mathbb{O}$, where \mathbb{O} is a pseudo-differential operator in the canonical form (4), therefore one can use (7). The consistency of eq. (9) with eq. (7) is provided by eq. (8).

Lemma 1.

$$D\overline{D}\partial^{-1}fD\overline{D}\partial^{-1} = (D\overline{D}\partial^{-1}f)D\overline{D}\partial^{-1} + D\overline{D}\partial^{-1}(f\overleftarrow{D\overline{D}\partial^{-1}}), \quad (10)$$

$$(\mathbb{O}_{\oplus}fD\overline{D}\partial^{-1})_{\ominus} = (\mathbb{O}_{\oplus}f)D\overline{D}\partial^{-1}, \quad (11)$$

$$(D\overline{D}\partial^{-1}f\mathbb{O}_{\oplus})_{\ominus} = D\overline{D}\partial^{-1}(f\overleftarrow{\mathbb{O}_{\oplus}}). \quad (12)$$

Proof. Equality (10) results from the following simple relation :

$$\partial^{-1}f\partial^{-1} = (\partial^{-1}f)\partial^{-1} + \partial^{-1}(f\overleftarrow{\partial^{-1}}) \quad (13)$$

One acts with $D\overline{D}$ on both sides of (13), then tries to push D and \overline{D} to the right in the first term, and to the left in the second term.

Equality (11) is obvious if one takes into account that the pseudo-differential operator $D\overline{D}\partial^{-1}$ being multiply either by D or \overline{D} from the right or left becomes a differential operator. Equality (12) is an operator-adjoint counterpart of equality (11). ■

It should be noted that, although there is an arbitrariness in the definition of the action of ∂^{-1} on a function f , this arbitrariness does not show up in (10) because it compensates between both terms on the right-hand side.

Proposition 2.

$$(L^p)_{\ominus} = \sum_{k=0}^{p-1} (L^{p-k-1}F)D\overline{D}\partial^{-1}(\overline{F} \overleftarrow{L^k}), \quad p \in \mathbb{N}. \quad (14)$$

Proof. The proof is by induction with the use of relations (10–12). Equation (14) is obviously correct at $p = 1$ (compare with eq. (1)). If it is correct for the $p = n$ case, then we have for the $p = n + 1$ case

$$(L^{n+1})_{\ominus} \equiv (LL^n)_{\ominus}$$

$$\begin{aligned}
&= \left(\left(I\partial + FD\bar{D}\partial^{-1}\bar{F} \right) \left((L^n)_\oplus + \sum_{k=0}^{n-1} (L^{p-k-1}F)D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k}) \right) \right)_\ominus \\
&= \left(FD\bar{D}\partial^{-1}\bar{F}(L^n)_\oplus + \sum_{k=0}^{n-1} (\partial L^{n-k-1}F)D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k}) \right. \\
&\quad \left. + FD\bar{D}\partial^{-1}\bar{F} \sum_{k=0}^{n-1} (L^{n-k-1}F)D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k}) \right)_\ominus \\
&= FD\bar{D}\partial^{-1} \left(\bar{F} \overleftarrow{(L^n)_\oplus} + \sum_{k=0}^{n-1} \bar{F}(L^{n-k-1}F) \overleftarrow{D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k})} \right) \\
&\quad + \sum_{k=0}^{n-1} \left((\partial + FD\bar{D}\partial^{-1}\bar{F})L^{n-k-1}F \right) D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k}) \\
&= FD\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^n}) + \sum_{k=0}^{n-1} (L^{n-k}F)D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k}) \\
&= \sum_{k=0}^n (L^{n-k}F)D\bar{D}\partial^{-1}(\bar{F} \overleftarrow{L^k}). \tag{15}
\end{aligned}$$

■

Proposition 3.

$$(L^{p+1})_\oplus = (L^p)_\oplus L - ((L^p)_\oplus F)D\bar{D}\partial^{-1}\bar{F} + \sum_{k=0}^{p-1} (L^{p-k-1}F)D\bar{D}(\bar{F} \overleftarrow{L^k}) \tag{16}$$

$$= L(L^p)_\oplus - FD\bar{D}\partial^{-1}(\bar{F} \overleftarrow{(L^p)_\oplus}) + \sum_{k=0}^{p-1} (L^{p-k-1}F)D\bar{D}(\bar{F} \overleftarrow{L^k}). \tag{17}$$

Proof. This is an easy calculation using eqs. (11,12,14) and obvious identities

$$\begin{aligned}
(L^{p+1})_\oplus &= (L^p)_\oplus L - ((L^p)_\oplus L_\ominus)_\ominus + ((L^p)_\ominus L_\oplus)_\oplus \\
&= L(L^p)_\oplus - (L_\ominus(L^p)_\oplus)_\ominus + (L_\oplus(L^p)_\ominus)_\oplus.
\end{aligned} \tag{18}$$

■

Corollary. Subtracting eq. (17) from eq. (16) we obtain

$$[(L^p)_\oplus, L] = ((L^p)_\oplus F)D\bar{D}\partial^{-1}\bar{F} - FD\bar{D}\partial^{-1}(\bar{F} \overleftarrow{(L^p)_\oplus}). \tag{19}$$

If one introduces evolution derivatives (flows) $\frac{\partial}{\partial t_p}$ according to the formula

$$\frac{\partial}{\partial t_p} F = ((L^p)_\oplus F), \quad \frac{\partial}{\partial t_p} \bar{F} = -(\bar{F} \overleftarrow{((L^p)_\oplus)}), \tag{20}$$

then eq. (19) takes the form of the Lax pair representation

$$\frac{\partial}{\partial t_p} L = [(L^p)_\oplus, L] \tag{21}$$

which was proposed in [1]. Actually, its self-consistency was proven in [1] only for the first few flows $p=0,1,2$ and 3, then conjectured for the general case there, and the corresponding integrable hierarchies were called the $N = 2$ supersymmetric unconstrained matrix $(k|n, m)$ -Generalized Nonlinear Schrödinger hierarchies. The algebra of the flows in (21) can easily be calculated

$$\left[\frac{\partial}{\partial t_m}, \frac{\partial}{\partial t_n}\right] = 0, \quad (22)$$

it is abelian algebra of the isospectral flows. The Lax-pair representation (21) may be seen as the integrability condition for the following linear system:

$$L\psi_1 = \lambda\psi_1, \quad (23)$$

$$\frac{\partial}{\partial t_p}\psi_1 = ((L^p)_{\oplus})\psi_1 \quad (24)$$

where λ is the spectral parameter and the eigenfunction ψ_1 is the Baker-Akhiezer function of the hierarchy.

Projecting the Lax-pair representation (21) on $D\bar{D}\partial^{-1}$, $D\partial^{-1}$, $\bar{D}\partial^{-1}$ and ∂^{-1} parts, one can straightforwardly extract the following evolution equations

$$\left(\frac{\partial}{\partial t_p}F\bar{F}\right) = (res(L^p))' + [res(L^p), F\bar{F}], \quad (25)$$

$$\left(\frac{\partial}{\partial t_p}F^*\bar{D}\bar{F}\right) = -(L^p)'_{D\partial^{-1}} + F\bar{F}(L^p)_{D\partial^{-1}} + res(L^p)(F^*\bar{D}\bar{F}), \quad (26)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t_p}F^*D\bar{F}\right) &= (L^p)'_{\bar{D}\partial^{-1}} + (L^p)_{\bar{D}\partial^{-1}}F\bar{F} - (F^*D\bar{F})res(L^p) \\ &\quad - F\bar{F}(Dres(L^p)) + res(L^p)(DF\bar{F}), \end{aligned} \quad (27)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t_p}FD\bar{D}\bar{F}\right) &= (L^p)'_{\partial^{-1}} + (F^*D\bar{F})(L^p)_{D\partial^{-1}} \\ &\quad + (L^p)_{\bar{D}\partial^{-1}}(F^*\bar{D}\bar{F}) + res(L^p)(DF^*\bar{D}\bar{F}) \end{aligned} \quad (28)$$

which can be used to express $res(L^p)$, $(L^p)_{D\partial^{-1}}$, $(L^p)_{\bar{D}\partial^{-1}}$ and $(L^p)_{\partial^{-1}}$, entering these equations, in terms of the time derivative $\frac{\partial}{\partial t_p}$ of different functionals of F and \bar{F} . With this aim we need to introduce a $k \times k$ matrix g by the consistent set of equations.

Definition 4.

$$g' = -gF\bar{F}, \quad (Dg) = -\left(\partial^{-1}g(DF\bar{F})g^{-1}\right)g, \quad (\bar{D}g) = -\left(\partial^{-1}g(\bar{D}F\bar{F})g^{-1}\right)g. \quad (29)$$

With the help of g the resolution of eqs. (25–28) with respect to $res(L^p)$, $(L^p)_{D\partial^{-1}}$, $(L^p)_{\bar{D}\partial^{-1}}$ and $(L^p)_{\partial^{-1}}$ is rather simple

$$res(L^p) = -(g^{-1}\frac{\partial}{\partial t_p}g) \equiv (\partial^{-1}\frac{\partial}{\partial t_p}F\bar{F}g^{-1})g, \quad (30)$$

$$\begin{aligned} (L^p)_{D\partial^{-1}} &= -g^{-1}(\partial^{-1}\frac{\partial}{\partial t_p}gF^*\bar{D}\bar{F}) \equiv -(\partial^{-1}\frac{\partial}{\partial t_p}F^*\bar{D}\bar{F}) \\ &\quad + (\partial^{-1}\frac{\partial}{\partial t_p}F\bar{F}g^{-1})(\partial^{-1}gF^*\bar{D}\bar{F}) - (\partial^{-1}\frac{\partial}{\partial t_p}F\bar{F}g^{-1}\partial^{-1}gF^*\bar{D}\bar{F}), \end{aligned} \quad (31)$$

$$(L^p)_{\bar{D}\partial^{-1}} = [(\partial^{-1}\frac{\partial}{\partial t_p}F^*D\bar{F}g^{-1})g], \quad (32)$$

$$\begin{aligned} (L^p)_{\partial^{-1}} &= (\partial^{-1}\frac{\partial}{\partial t_p}FD\bar{D}\bar{F}) - [(\partial^{-1}\frac{\partial}{\partial t_p}F^*D\bar{F}g^{-1})(\partial^{-1}gF^*\bar{D}\bar{F})] \\ &\quad + (\partial^{-1}\frac{\partial}{\partial t_p}F^*D\bar{F}g^{-1}\partial^{-1}gF^*\bar{D}\bar{F}) \end{aligned} \quad (33)$$

where in eqs. (32–33) the fermionic derivative D entering the square brackets acts on the right inside these brackets. This can easily be verified by directly substituting these expressions into the original equations (25–28) and using eqs. (29).

Proposition 4.

$$\begin{aligned} & (L^{p+1})_{\oplus} \\ &= (L^p)_{\oplus} L - \left(\frac{\partial}{\partial t_p} F\right) D \bar{D} \partial^{-1} \bar{F} + \left(\partial^{-1} \frac{\partial}{\partial t_p} F D \bar{D} \bar{F}\right) \\ & - \left(\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1}\right) (\partial^{-1} g F^* \bar{D} \bar{F}) + \left(\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1} \partial^{-1} g F^* \bar{D} \bar{F}\right) \end{aligned} \quad (34)$$

$$\begin{aligned} &= L(L^p)_{\oplus} + F D \bar{D} \partial^{-1} \left(\frac{\partial}{\partial t_p} \bar{F}\right) + \left(\partial^{-1} \frac{\partial}{\partial t_p} F D \bar{D} \bar{F}\right) \\ & - \left(\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1}\right) (\partial^{-1} g F^* \bar{D} \bar{F}) + \left(\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1} \partial^{-1} g F^* \bar{D} \bar{F}\right) \end{aligned} \quad (35)$$

where in these equations the fermionic derivatives D and \bar{D} entering the brackets act as operators on the right both inside and outside the brackets.

Proof. Taking $(L^{p+1})_{\oplus}$ in (16) and using eqs. (20) as well as the identity

$$\sum_{k=0}^{p-1} (L^{p-k-1} F) D \bar{D} (\bar{F} \overleftarrow{L^k}) = \text{res}(L^p) D \bar{D} + (L^p)_{D\partial^{-1}} D + (L^p)_{\bar{D}\partial^{-1}} \bar{D} + (L^p)_{\partial^{-1}} \quad (36)$$

which follows from eq. (14), one can easily obtain the following expression:

$$\begin{aligned} (L^{p+1})_{\oplus} &= ((L^p)_{\oplus}) L - \left(\frac{\partial}{\partial t_p} F\right) D \bar{D} \partial^{-1} \bar{F} + \text{res}(L^p) D \bar{D} \\ &+ (L^p)_{D\partial^{-1}} D + (L^p)_{\bar{D}\partial^{-1}} \bar{D} + (L^p)_{\partial^{-1}}. \end{aligned} \quad (37)$$

Substituting $\text{res}(L^p)$ (30), $(L^p)_{D\partial^{-1}}$ (31), $(L^p)_{\bar{D}\partial^{-1}}$ (32) and $(L^p)_{\partial^{-1}}$ (33) into eq. (37), we arrive at the first equality (34). The second equality (35) can obviously be derived from the first one if one substitutes $(L^p)_{\oplus} L$ by $L(L^p)_{\oplus} + \frac{\partial}{\partial t_p} L$ there, according to eq. (21). ■

Corollary: recursion relations. Applying the noncommutatively-adjoint of operator relation (34) to the supermatrix valued superfield \bar{F} from the right and similarly applying (35) to F from the left as well as using eqs. (20) it is not complicated to obtain recurrence relations relating flows with the evolution derivatives $\frac{\partial}{\partial t_{p+1}}$ and $\frac{\partial}{\partial t_p}$

$$\begin{aligned} \left(\frac{\partial}{\partial t_{p+1}} \bar{F}\right) &= -\left(\frac{\partial}{\partial t_p} \bar{F}\right)' + (\bar{D} D \partial^{-1} \frac{\partial}{\partial t_p} \bar{F} F) \bar{F} - [\bar{F} (\partial^{-1} \frac{\partial}{\partial t_p} F \overleftarrow{D} \bar{D} \bar{F}) \\ &- \bar{F} (\partial^{-1} \frac{\partial}{\partial t_p} F^* \overleftarrow{D} \bar{F} g^{-1}) (\partial^{-1} g F^* \overleftarrow{D} \bar{F}) + \bar{F} (\partial^{-1} \frac{\partial}{\partial t_p} F^* \overleftarrow{D} \bar{F} g^{-1} \partial^{-1} g F^* \overleftarrow{D} \bar{F})], \end{aligned} \quad (38)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t_{p+1}} F\right) &= \left(\frac{\partial}{\partial t_p} F\right)' + (F D \bar{D} \partial^{-1} \frac{\partial}{\partial t_p} \bar{F} F) + [(\partial^{-1} \frac{\partial}{\partial t_p} F D \bar{D} \bar{F}) F \\ &- (\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1}) (\partial^{-1} g F^* \bar{D} \bar{F}) F + (\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1} \partial^{-1} g F^* \bar{D} \bar{F}) F] \end{aligned} \quad (39)$$

where the fermionic derivatives D and \bar{D} , entering the square brackets in eqs. (38) and (39), act inside these brackets on the left and right, respectively.

3 Super–algebraic approach

Following the super–algebraic approach⁴ of ref. [10], in [2] a wide class of integrable hierarchies was constructed which corresponds to the homogeneous gradation of the loop superalgebra $sl(2k+n|2k+m) \otimes C[\lambda, \lambda^{-1}]$ with the splitting matrix E and the grading operator d ,

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad d = \lambda \frac{\partial}{\partial \lambda}. \quad (40)$$

The corresponding isospectral flows are [2]

$$\left(\frac{\partial}{\partial t_p} \mathcal{L}_z \right) = [(\Theta \lambda^p E \Theta^{-1})_+ - (G^{-1} \frac{\partial}{\partial t_p} G), \mathcal{L}_z] \quad (41)$$

where the dressing matrix Θ is obtained from dressing the Lax operator \mathcal{L}_z

$$\mathcal{L}_z := \partial - \lambda E + \mathcal{A} = \Theta^{-1} (\partial - \lambda E) \Theta, \quad \Theta = 1 + \sum_{k=1}^{\infty} \lambda^{-k} \theta^{(-k)}. \quad (42)$$

Hereafter, the subscript $+$ denotes the projection on the positive homogeneous grading (40),

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & F & 0 & 0 \\ 0 & 0 & (DF) & 0 & 0 \\ -(D\overline{D} \overline{F}) & (\overline{D} \overline{F}^*) & 0 & -(D\overline{F}^*) & -\overline{F} \\ -(F^* \overline{D} \overline{F}) & 0 & (\overline{D} F) & -F\overline{F} & 0 \\ -(DF^* \overline{D} \overline{F}) & (F^* \overline{D} \overline{F}) & (D\overline{D} F) & -(DF\overline{F}) & -F\overline{F} \end{pmatrix} \quad (43)$$

and

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -(\partial^{-1} g F^* \overline{D} \overline{F}) & 0 & 0 & g & 0 \\ -(D\partial^{-1} g F^* \overline{D} \overline{F}) & (\partial^{-1} g F^* \overline{D} \overline{F}) & 0 & (Dg) & g \end{pmatrix}. \quad (44)$$

It is easily seen that the matrix G (44) entering into the Lax–pair representation (41) is nonlocal. Moreover, the $N = 2$ superfield entries of the connection \mathcal{A} (43) are not independent quantities, i.e. they are subjected to constraints. Why in this case do isospectral matrix flows (41) be local, as it is obviously the case for the flows (21)? Why are they supersymmetric, or in other words, why do these flows preserve the above–mentioned constraints? Finally, how can one see in general that these flows coincide with the isospectral flows (21). These questions were raised in [2], but clarified only partly there. Based on the pseudo–differential approach

⁴For more recent development of the super–algebraic approach, see ref. [11] and references therein.

developed in the previous Section we are able to prove here that the super-algebraic isospectral matrix flows (41) are equivalent to the pseudo-differential isospectral flows (21), therefore the former are local and supersymmetric as well, because it is the case for the latter.

The Lax-pair representation (41) may be seen as the integrability condition for the following linear system:

$$\mathcal{L}_z \Psi = 0, \quad (45)$$

$$\left(\frac{\partial}{\partial t_p} G \Psi\right) = (G \Theta \lambda^p E(G \Theta)^{-1})_+ G \Psi \quad (46)$$

where $\Psi^T = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$. In order to prove equivalence of the Lax-pair representations (41) and (21) it is enough to prove equivalence of the corresponding linear systems (45–46) and (23–24).

Proposition 5. The linear systems (45–46) and (23–24) are equivalent.

Proof. The first equation (45) of the linear system (45–46) is equivalent to the first equation (23) of the linear system (23–24) and possesses the solution

$$\Psi = \begin{pmatrix} \psi_1 \\ (D\psi_1) \\ (D\bar{D}\partial^{-1}\bar{F}\psi_1) \\ (\bar{D}\psi_1) \\ (D\bar{D}\psi_1) \end{pmatrix} \quad (47)$$

which was actually observed in [2], and it was the starting point for the super-algebraic construction developed there.

In order to demonstrate that the second equation (46) of the linear system (45–46) is equivalent to the second equation (24) of the linear system (23–24) as well, we use the equality [10]

$$(G \Theta \lambda^{p+1} E(G \Theta)^{-1})_+ = \lambda(G \Theta \lambda^p E(G \Theta)^{-1})_+ + \left(\frac{\partial}{\partial t_p} G \theta^{(-1)}\right) \quad (48)$$

and rewrite eq. (46) in the following equivalent form:

$$\left(\frac{\partial}{\partial t_{p+1}} G \Psi\right) = \lambda \left(\frac{\partial}{\partial t_p} G \Psi\right) + \left(\frac{\partial}{\partial t_p} G \theta^{(-1)}\right) G \Psi. \quad (49)$$

Then, substituting G (44), Ψ (47), and $\theta^{(-1)}$, derived from the dressing condition (42), into eq. (49), the latter becomes

$$\begin{aligned} \left(\frac{\partial}{\partial t_{p+1}} \psi_1\right) &= \lambda \left(\frac{\partial}{\partial t_p} \psi_1\right) + [((\partial^{-1} \frac{\partial}{\partial t_p} F D \bar{D} \bar{F}) - (\frac{\partial}{\partial t_p} F) D \bar{D} \partial^{-1} \bar{F}) \\ &\quad - (\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1})(\partial^{-1} g F^* \bar{D} \bar{F}) + (\partial^{-1} \frac{\partial}{\partial t_p} F^* D \bar{F} g^{-1} \partial^{-1} g F^* \bar{D} \bar{F})] \psi_1 \end{aligned} \quad (50)$$

where the fermionic derivatives D and \bar{D} , entering the square brackets, act inside these brackets. Eq. (50) is satisfied if and only if eq. (24) is satisfied, and the latter is obvious if one takes into account eq. (23) and relation (34) of the Proposition 4. ■

Let us discuss shortly the locality of the isospectral flows in the super-algebraic Lax-pair representation (41). The connection \mathcal{A} (43) entering into the Lax operator \mathcal{L} (42) is a local functional of the supermatrix-valued superfields F , \overline{F} and their derivatives. It is also known [10] that the matrix $(\Theta \lambda^p E \Theta^{-1})_+$ is a local functional. Using (44), (30–31) one can calculate the second term of the Lax representation (41)

$$(G^{-1} \frac{\partial}{\partial t_p} G) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (L^p)_{D\partial^{-1}} & 0 & 0 & -res(L^p) & 0 \\ (D(L^p)_{D\partial^{-1}}) & -(L^p)_{D\partial^{-1}} & 0 & -(D res(L^p)) & -res(L^p) \end{pmatrix} \quad (51)$$

which is a local functional as well. Thus, all the objects involved into the Lax-pair representation (41) are local, therefore the same is true with respect to the corresponding isospectral flows.

Acknowledgments. A.S. would like to thank the Laboratoire de Physique de l'ENS Lyon and Institut für Theoretische Physik, Universität Hannover, for the hospitality during the course of this work. This work was partially supported by the RFBR Grant No. 06-01-00627-a, RFBR-DFG Grant No. 06-02-04012-a, DFG Grant 436 RUS 113/669-3, the Program for Supporting Leading Scientific Schools (Grant No. NSh-5332.2006.2), and by the Heisenberg-Landau Program.

References

- [1] A.S. Sorin and P.H.M. Kersten, *The $N=2$ supersymmetric unconstrained matrix GNLS hierarchies*, Lett. Math. Phys. **60** (2002) 135, nlin.SI/0201026.
- [2] F. Delduc and A.S. Sorin, *Recursion operators of the $N=2$ supersymmetric unconstrained matrix GNLS hierarchies*, JHEP Proceedings, PrHEP unesp2002, Workshop on Integrable Theories, Solitons and Duality, 1-6 July 2002, Sao Paulo, Brazil, nlin.SI/0206037.
- [3] L. Bonora, S. Krivonos, and A. Sorin, *The $N = 2$ supersymmetric matrix GNLS hierarchies*, Lett. Math. Phys. **45** (1998) 63, solv-int/9711009.
- [4] L. Bonora, S. Krivonos, and A. Sorin, *Coset approach to the $N = 2$ supersymmetric matrix GNLS hierarchies*, Phys. Lett. **A240** (1998) 201, solv-int/9711012.
- [5] L. Bonora, S. Krivonos, and A. Sorin, *Towards the construction of $N = 2$ supersymmetric integrable hierarchies*, Nucl. Phys. **B477** (1996) 835, hep-th/9604165.
- [6] L. Bonora and A. Sorin, *The Hamiltonian structure of the $N=2$ supersymmetric GNLS hierarchy*, Phys. Lett. **B407** (1997) 131, hep-th/9704130.

- [7] Z. Popowicz, *The extended supersymmetrization of the multicomponent Kadomtsev-Petviashvili hierarchy*, J. Phys. **A29** (1996) 1281, hep-th/9510185.
- [8] A.P. Fordy and P.P. Kulish, *Nonlinear Schrödinger equations and simple Lie algebras*, Commun. Math. Phys. **89** (1983) 427.
- [9] P.H.M. Kersten and A.S. Sorin, *Bi-Hamiltonian structure of the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy*, Phys. Lett. **A300** (2002) 397, nlin.SI/0201061.
- [10] H. Aratyn, J.F. Gomes, E. Nissimov, S. Pacheva, and A.H. Zimerman, *Symmetry Flows, Conservation Laws and Dressing Approach to the Integrable Models*, in *Integrable Hierarchies and Modern Physical Theories*, Eds. H. Aratyn and A.S. Sorin, Kluwer Acad. Publ., Dordrecht/Boston/London, 2001, pg. 243, nlin.SI/0012042.
- [11] H. Aratyn, J.F. Gomes, G.M. de Castro, M.B. Silka, and A.H. Zimerman, *Supersymmetry for integrable hierarchies on loop superalgebras*, J. Phys. **A38** (2005) 9341, hep-th/0508008.